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Impartial Games

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In memory of Jack Kenyon, 1935-08-26 to 1994-09-19

Abstract. We give examples and some general results about impartial games, those in which both players are allowed the same moves at any given time.

1. Introduction

We continue our introduction to combinatorial games with a survey of impartial games. Most of this material can also be found in WW [Berlekamp et al. 1982], particularly pp. 81–116, and in ONAG [Conway 1976], particularly pp. 112–130. An elementary introduction is given in [Guy 1989]; see also [Fraenkel 1996], pp. **??**–**??** in this volume.

An *impartial game* is one in which the set of Left options is the same as the set of Right options. We've noticed in the preceding article the impartial games

$$\{\ \ |\ \ \} = *0 = 0, \qquad \{0\,|\,0\} = *1 = * \quad \text{and} \quad \{0, *\,|\,0, *\} = *2.$$

that were born on days 0, 1, and 2, respectively, so it should come as no surprise that on day n the game

$$*n = \{*0, *1, *2, \ldots, *(n{-}1) \mid *0, *1, *2, \ldots, *(n{-}1)\}$$

is born. In fact any game of the type

$$\{*a, *b, *c, \ldots \mid *a, *b, *c, \ldots\}$$

has value $*m$, where $m = \text{mex}\{a, b, c, \ldots\}$, the least nonnegative integer *not* in the set $\{a, b, c, \ldots\}$. To see this, notice that any option, $*a$ say, for which $a > m$,

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is reversible, both as a Left option and as a Right option, because $*m$ is an option of $*a$,

$$*m \text{ is both } \geq \text{ and } \leq \{*a, *b, *c, \ldots \,|*a, *b, *c, \ldots\}$$

so that $*a$ may be replaced by the options of $*m$, namely 0, $*$, $*2$, …, $*(m-1)$.

This is the inductive step that proves the Sprague–Grundy theorem [Sprague 1935–36; Grundy 1939], which states that every position in an impartial game (or, which is the same, every impartial game) is equivalent to a *nim-heap* (see page **??** ff. in Fraenkel's article in this volume).

SInce the Left and Right options are the same, $*n$ is its own negative, $*n + *n = 0$. Also, we need only write one set of options, and may define the *nimber*

$$*n = \{0, *, *2, \ldots, *(n-1)\}.$$

This exactly parallels John von Neumann's definition of ordinal numbers.

2. Examples of Impartial Games

We all know that the game of *Nim* is played with several heaps of beans. A move is to select a heap, and to remove any positive number of beans from it, possibly the whole heap. Any position in Nim is therefore the sum of several one-heap Nim games. The value of a single heap of n beans is $*n$.

It's easy to see how to win a game of Nim if there's only one (nonempty) heap: take the whole heap! But it's worth pausing for a moment to note exactly what your options are. They are to move to *any* smaller sized heap: they correspond exactly to the options in the definition of $*n$. It's also fairly easy to play well at two-heap Nim: if the heaps are unequal in size, remove enough beans from the larger to make the heaps equal. If the two heaps are already equal, then hope that it is your opponent's turn to move. From then on, use the *Tweedledum and Tweedledee Principle*: whatever your opponent does to one heap, you copy in the other. For more than two heaps, the theory is more tricky. It was discovered by Bouton [1902]. Imagine each heap to be partitioned into distinct powers of two. For example, Figure 1 shows four heaps of 27, 23, 22 and 15 beans partitioned in this way.

We can pair off and then ignore heaps of equal size, so concentrate on the columns with an *odd* number of parts, the ones, fours and sixteens. A good move would be to take $16 + 4 + 1 = 21$ from the 23 heap. If you leave a position for your opponent in which each power of two occurs *evenly* often, he will have to change the parity in at least one column, and then you will be able to restore it. Notice that finding a good move does *not* depend on there being appropriate powers of two all in the same row (heap). You could also take $16 - 4 + 1 = 13$ from the 27 heap, or $16 + 4 - 1 = 19$ from the 22 heap. Find three good moves from the Nim position {23, 19, 13, 12, 11}.

Figure 1. How to look at a Nim position.

Figure 2. A game of Nimble.

Nimble is played with coins on a strip of squares (Figure 2). Take turns, moving just one coin to the left. You can jump onto or over other coins, even clear off the strip. You can have any number of coins on a square. The last player wins. Can you analyze this game? Suppose that you are not allowed to jump off the strip, so that the game ends when all coins are stacked on the left hand square. Can your analysis be modified to cope with this variant?

In the game shown in Figure 3, you're allowed at most one coin on a square, and you're not allowed to jump over other coins. A move is to *slide* a coin leftwards as far as you like, but not onto or over the next coin, and not off the end of the strip. The analysis is now more cunning: the black marks on the side of the strip may give you a hint.

Figure 3. A coin-sliding game.

Figure 4. De Bruijn's Silver Dollar Game.

Figure 4 shows N. G. de Bruijn's *Silver Dollar Game*, which is played like the previous game, but one coin is worth much more than all the others put together, the leftmost square is replaced by a money-bag, and there's the additional option of taking the money-bag. If you do this, your opponent gets the coins left on the strip. When you've solved that game, consider the variant in which the additional option is to slide a coin *and* take the money-bag, all in one move.

Poker Nim is played like Nim, but with poker chips in place of beans; as well as removing chips from a heap, you may instead *add* chips to a heap. How does this affect play?

In *Northcott's Game* there is one checker of each color on each row of a checkerboard (Figure 5). A move is to slide one of your checkers any number of squares in its own row, without jumping over your opponent's checker and without going off the board. This *looks* like a *partizan* game, and many people can't see any

Figure 5. Northcott's Game.

point to it, and slide the checkers aimlessly. Indeed, the game doesn't appear to satisfy the *ending condition*, but there is a winner, and she can force the game to end. Remember that the aim is to be the last player to move.

Lasker's Nim is played like ordinary Nim, but with the additional option that you are allowed to split a heap into two smaller, nonempty heaps.

3. Nim-Addition

(See ONAG, pp. 50–51; WW, pp. 60–61.) We know that $*n+0 = *n$ and that $*n + *n = 0$, and it's not hard to see that addition of impartial games, indeed of any of our games, is commutative and associative. Let's calculate

$$*2+* = \{0, *\}+\{0\} = \{0+*, *+*, *2+0\} = \{*, 0, *2\} = *3.$$

Add $*$, or $*2$, to each side and obtain $*2 = *3 + *$ and $* = *3 + *2$. In general,

$$\begin{aligned} *a+*b &= \{0, *, *2, \ldots, *(a-1)\}+\{0, *, *2, \ldots, *(b-1)\} \\ &= \{0+*b, *+*b, \ldots, *(a-1)+*b, *a+0, *a+*, \ldots, *a+*(b-1)\}, \end{aligned}$$

and we can build a nim-addition table (Table 1) by noting that the options of an entry are just the earlier entries in the same row and the earlier entries in the same column. Each entry in Table 1 is the least nonnegative integer not appearing as an earlier entry in the same row or column. For instance, $*5 + *6 = *3$, because 3 is the first number not in the set $\{5, 4, 7, 6, 1, 0, 6, 7, 4, 5, 2\}$, i.e., the first six entries in row 5 and the first five entries in column 6. In the usual language, 3 is the *nim-sum* of 5 and 6, which is sometimes written $5 \overset{*}{+} 6 = 3$.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 1. Nim-addition table. The stars have been omitted; i.e., the entries are nim-values instead of nimbers.

Contrast the two equations $*5 + *6 = *3$ and $5 \overset{*}{+} 6 = 3$. In the first the summands are nimbers, i.e., values of impartial games, and the addition is a game sum. In the second the summands are *nim-values* and the addition is nim-addition.

Nim-addition is perhaps better known as addition without carry in base 2, or vector or coordinatewise addition over GF(2), or XOR (exclusive or): it is reassuring that it also follows from the more general idea of game sum.

Many of the games mentioned in the previous section are disguises for Nim, often with the addition of *reversible moves*, which we mentioned in the preceding article. As we saw in the Introduction, every impartial game is equivalent to a *bogus nim-heap*, i.e., a heap of m beans ("m" for "mex"), together with some (reversible) options that *increase* the size of the heap.

To summarize the Sprague-Grundy theory: the nim-value of the sum of two impartial games is the nim-sum of their separate nim-values. Impartial games belong to one of only two outcome classes: all positions are either

$\mathcal{P}$-positions	previous-player-winning	nim-value zero, or
$\mathcal{N}$-positions	next-player-winning	nonzero nim-value.

In the literature, $\mathcal{P}$-positions are sometimes called "safe" or "good" or "winning" without indicating which player enjoys this happy situation.

4. Subtraction Games

(See WW, pp. 83–86, 487–498.) Subtraction games are very simple examples of impartial games, played, like Nim, with heaps of beans. A move in the game $S(s_1, s_2, s_3, \ldots)$ is to take a number of beans from a heap, provided that number is a member of the *subtraction-set*, $\{s_1, s_2, s_3, \ldots\}$. Analysis of such a game and of many other heap games is conveniently recorded by a *nim-sequence*,

$$n_0 n_1 n_2 n_3 \ldots,$$

meaning that the nim-value of a heap of h beans is n_h, $h = 0, 1, 2, \ldots$, i.e., that the value of a heap of h beans in this particular game is the nimber $*n_h$. In this section, and often later, to avoid printing stars, we say that the nim-value of a position is n, meaning that its value is the nimber $*n$.

Table 2 shows some examples: the first is a manifestation of She-Loves-Me-She-Loves-Me-Not; the last is Nim. If the subtraction-set is finite, the nim-sequence is (ultimately) periodic. But little is known about the length of the period *vis à vis* the membership of the subtraction set.

In subtraction games the nim-values 0 and 1 are remarkably related by *Ferguson's Pairing Property* [Ferguson 1974; WW, pp. 86, 422]: if s_1 is the least member of the subtraction-set, then

$$\mathcal{G}(n) = 1 \qquad \text{just if} \qquad \mathcal{G}(n - s_1) = 0.$$

Subtraction game	Nim-sequence	(ultimate)	Period
$S(1)$	$\dot{0}\dot{1}01010101\ldots$		2
$S(2)$	$\dot{0}01\dot{1}0011001100\ldots$		4
$S(3)$	$\dot{0}0011\dot{1}0001110001110\ldots$		6
$S(1,2)$	$\dot{0}1\dot{2}0120120120\ldots$		3
$S(1,2,3)$	$\dot{0}12\dot{3}0123012301230\ldots$		4
$S(1,2,3,4)$	$\dot{0}123\dot{4}0123401234012340\ldots$		5
$S(2,4,7)$	$00112203\dot{1}0\dot{2}10210210\ldots$		3
$S(2,5,6)$	$\dot{0}011021302\dot{1}0011021302100\ldots$		11
$S(4,10,12)$	$\dot{0}00011110022113300221\dot{1}0000\ldots$		22
$S(1,2,3,4,\ldots)$	$0123456789\ldots$	(saltus 1 and)	1

Table 2. Nim-sequences and periods for subtraction games.

Here and later "$\mathcal{G}(n) = v$" means that the nim-value of a heap of n beans is v.

5. Take-and-Break Games

(See WW, pp. 81–106.) Guy and Smith [1956] devised a code classifying a broad range of impartial games played with heaps or rows. Suppose a game has code

$$\mathbf{d}_0{\cdot}\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\ldots,$$

where the *code digits* $\mathbf{d}_k$ are nonnegative integers. If the binary expansion of $\mathbf{d}_k$ is

$$\mathbf{d}_k = 2^{a_k} + 2^{b_k} + 2^{c_k} + \cdots,$$

where $0 \le a_k < b_k < c_k < \cdots$, then it is legal to remove k beans from a heap, provided that the rest of the heap is left in exactly a_k or b_k or c_k or $\ldots$ nonempty heaps.

In order that the game should satisfy the ending condition, $\mathbf{d}_0$ must be divisible by 4, i.e., $a_0 \ge 2$.

Subtraction games are the special case $\mathbf{d}_s = \mathbf{3}$ when s is in the subtraction-set, and $\mathbf{d}_k = \mathbf{0}$ otherwise.

Octal games are those with code digits $\mathbf{d}_k \le \mathbf{7}$ for all k. Guy and Smith showed that an octal game is *ultimately periodic* with period p, i.e.,

$$\mathcal{G}(n+p) = \mathcal{G}(n) \quad \text{for all} \quad n > n_0 = 2e + p + t,$$

provided that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for $n \le n_0$ apart from some exceptional values of n, of which e is the largest, and $\mathbf{d}_k = 0$ for $k > t$, i.e., the maximum number of beans that may be taken from a heap in a single move is t. Whether all such finite octal games are ultimately periodic remains a difficult open question. They cannot be *arithmetically periodic*: that is, there is no period p and *saltus* $s > 0$, such that $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for all large enough n (WW, p. 114).

Table 3 exhibits some specimen games, of which the last three are *hexadecimal games* with $\mathbf{d}_k \le \mathbf{15} = \mathbf{F}$. Such games may be arithmetically periodic. Anil Gangolli and Thane Plambeck established the ultimate periodicity of four octal games that were previously unknown:

The game **.16** has period 149459 (a prime!), the last exceptional value being $\mathcal{G}(105350) = 16$. The game **.56** has period 144 and last exceptional value $\mathcal{G}(326639) = 26$. The games **.127** and **.376** each have period 4 (with cycles of values 4, 7, 2, 1 and 17, 33, 16, 32 respectively) and last exceptional values $\mathcal{G}(46577) = 11$ and $\mathcal{G}(2268247) = 42$.

Grundy's Game [Grundy 1939; WW, p. 111], in which the move is to split a heap into two *unequal* heaps, continues to defy complete analysis, despite Mike Guy's calculation of the first ten million nim-values. Among these values,

$$0, 1, 6, 7, 10, 11, 12, 13, 18, 19, 20, 21, 24, \ldots$$

occur quite rarely. When written in binary, these values contain an even number of ones *if you ignore the last digit.* These *rare* values form a closed space (the *sparse space*) under nim-addition, while the complement forms the *common coset*:

$$\begin{array}{ccccccccc} \text{rare} & \overset{*}{+} & \text{rare} & = & \text{rare} & = & \text{common} & \overset{*}{+} & \text{common} \\ \text{rare} & \overset{*}{+} & \text{common} & = & \text{common} & = & \text{common} & \overset{*}{+} & \text{rare} \end{array}$$

If the nim-values in a sequence begin to cluster in a suitable common coset, this clustering is likely to persist. In Kayles the rare and common values are *evil* and *odious* numbers respectively, with an even and odd number of ones in their binary expansions. On the other hand, Dawson's Kayles doesn't exhibit this sparse space phenomenon. In Grundy's Game only 1273 rare values have appeared; the only one in the range $36184 < n \le 10^7$ is $\mathcal{G}(82860) = 108$. If the rare values have indeed died out, then Grundy's Game will ultimately be periodic, but the period may be astronomical.

Amongst the comparative chaos, John Conway and Mike Guy have noted a remarkable structure in the nim-values for Grundy's Game, related to the number 59. The probability that $\mathcal{G}(n+d) = \mathcal{G}(n)$ is often as high as $\frac{1}{4}$ if

$$d \quad \text{is } \textit{near} \quad 59k \quad \text{and} \quad d \equiv k \bmod 3.$$

Examples of these pseudo-periods are 58, 61, 116, 119, 122, 290, 293, 296, 360, 412, 580, 583, 586, 589, 647, 650, 882, 952, 1172, where the last four correspond to $k = 11, 15, 16, 20$.

6. Green Hackenbush

(See ONAG, pp. 165–172; WW, pp. 183–190.) This is played on a picture, as in Blue-Red Hackenbush, but now all the edges are grEen, and may be chopped by Either player, making it an impartial game. Every Green Hackenbush picture has a nim-value: for example (Figure 6, right) the value of a string of 6 green

Code	*Game*	*Nim-sequence*
·77	*Kayles.* Knock down one skittle, or two contiguous skittles, from a row. [Dudeney 1908, Loyd 1914].	Ultimate period $p = 12$, $\dot{4}12814721827\dot{7}$ except for $n = 0, 3, 6, 9, 11, 15, 18, 21, 22, 28,$ 34,39,57,70, nim-value is resp. 0,3,3,4, 6,7,3,4,6,5,6,3,4,6.
·137	*Dawson's Chess.* $3 \times n$ board. White and Black pawns on ranks 1 and 3. Capturing obligatory. Looks partizan but isn't. [Dawson 1934; 1935].	$\dot{8}11203110332244559330113021104537\dot{4}$ except 0 for $n = 0, 14, 34$ and 2 for $n = 16, 17, 51$. $p = 34$.
·07	*Dawson's Kayles.* Knock down 2 skittles, but only if they're contiguous.	As for ·**137**, but shifted one term: 0011203. . . in place of 011203. . .
·6	*Officers.* Take 1 counter from any *longer* row. [Descartes 1953].	No period found. $\mathcal{G}(10342) = 256$.
·007	*Treblecross.* One-dimensional tic-tac-toe. (WW, pp. 93–94).	No pattern yet found.
·077	*Duplicate Kayles.* Knock down 2 or 3 contiguous skittles [Guy and Smith 1956].	$p = 24$. Kayles with each nim-value repeated, 00112233114433. . .
·7777	*Double Kayles.* Take up to 4 beans from a heap; leave rest in at most 2 heaps. [Guy and Smith 1956; WW, p. 98].	$p = 24$. Kayles with each nim-value g replaced by the pair $2g, 2g + 1$ or $2g + 1, 2g$ (according to a certain rule), 0123456732897654328945. . .
·156	See [Kenyon 1967].	$p = 349$.
·165	See [Austin 1976].	$p = 1550$.
4.3	*Lasker's Nim*	0124356879. . . $p = s = 4$.
·8	(first cousin of) *Triplicate Nim.* Take 1 from heap, rest left in exactly 3 nonempty heaps.	Arithmetically periodic, $p = 3$, saltus = 1. 0000111222333444. . .
·3F	(F=15) *Kenyon's Game.* Take 1 from heap or take 2 and leave rest in any number of heaps up to 3. [Kenyon 1967].	$p = 6$, $s = 3$ 0120123453456786789. . .
·E	(E =14) Take 1, leave rest in just 1, 2 or 3 heaps.	001234153215826514. . . $\mathcal{G}(246) = 128$. No known pattern.

Table 3. Some sample take-and-break games.

edges is $*6$. It is clear that the six possible moves exactly parallel the six possible moves that you can make from a heap of six beans.

We will see how to evaluate Green Hackenbush trees by the Colon Principle and how to reduce any picture to a forest by the Fusion Principle.

Green Hackenbush trees are examples of the *ordinal sum* $G : H$, which can be defined [WW, p. 214] for any two games G and H

$$G : H = \{G^L,\, G : H^L \,|\, G^R,\, G : H^R\}$$

where any move in G annihilates H, while a move in H leaves G unaffected. The *Colon Principle* [WW, pp. 184–185] states that $H \geq K$ implies $G : H \geq G : K$, and, in particular, that $H = K$ implies $G : H = G : K$. That is, $G : H$ depends only on the *value* of H and not on its *form*. It *may* depend on the *form* of G, because there are games $G_1 = G_2$ for which $G_1 : H \neq G_2 : H$.

The Colon Principle applies at branch points of Green Hackenbush trees, allowing us to do nim-addition "up in the air." For example, at a in Figure 6, left, we have $*3 + *2 = *$; at b, $* + * = 0$; and at c, $* + *2 = *3$, so the value is the same as that of Figure 6, middle, where, at d, $*2 + *2 + * + *4 = *5$, and the tree is worth $*6$. Notice the interplay of ordinary addition along strings, with nim-addition at branch points.

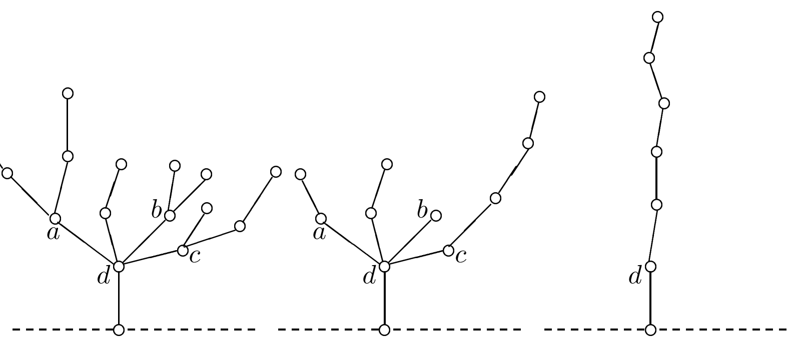

Figure 6. Transforming a tree into a stalk.

Green Hackenbush pictures involving circuits can be evaluated by the *Fusion Principle* (WW, pp. 186–188):

> The value of a picture is unaltered
> if you identify the nodes of a circuit.

The edges of the circuit then become loops, which may be replaced by twigs: compare Figure 7, middle and right. Check that the value of Figure 7, left, is $*8$. In this way, every component of a Green Hackenbush picture can be reduced to a tree, and hence to a string, and the strings are combined by nim-addition.

7. Welter's Game

(See [Welter 1952; 1954; Berlekamp 1972]; ONAG, pp. 153–165; WW, pp. 472–481.) This is another game whose analysis involves the interplay of nim-addition and ordinary addition. It is a form of Nim with unequal heaps, but in order to

Figure 7. Girl becomes tree.

keep track of empty heaps, only one of which is allowed, it's better to play it with coins on a strip of squares, numbered 0, 1, 2, ..., with at most one coin on a square. A move is to shift a coin leftwards to any unoccupied square, possibly passing over other coins. The game ends when the k coins are on the leftmost squares 0, 1, ..., $k-1$. Figure 8 shows a position with $k = 7$.

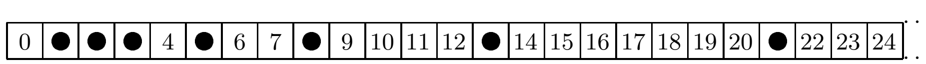

Figure 8. The position {1,2,3,5,8,13,21} in Welter's Game.

To calculate the nim-value, or *Welter function*, $[a|b|c|\ldots]_k$ of the position with k coins on squares a, b, c, ..., first note that for $k = 1$, $[a] = a$, and that for $k = 2$, $[a|b]$ is one less than the nim-sum of a and b: e.g., $[1|3] = 1$, $[5|6] = 2$. For more than two coins, *mate* the pair that is congruent modulo the highest power of two (it doesn't matter that this pair may not be unique). Remove the mated pair and find the best mated pair among the remaining $k-2$ coins. Continue until all coins are mated, except, when k is odd, for one coin, the *spinster*, s. Then, if (a,b), (c,d), ... are the mates, $[a|b|c|d|\ldots]$ may be calculated as the *nim-sum*

$$[a|b] \overset{*}{+} [c|d] \overset{*}{+} \ldots (\overset{*}{+} [s])$$

where the last term is included just if k is odd.

In Figure 8 the best mates are (5,21), then (1,13), then (2,8), and 3 is the spinster, so the nim-value is

$$\begin{aligned}[1|2|3|5|8|13|21] &= [5|21] \overset{*}{+} [1|13] \overset{*}{+} [2|8] \overset{*}{+} 3 \\ &= 15 \overset{*}{+} 11 \overset{*}{+} 9 \overset{*}{+} 3 \;=\; 14.\end{aligned}$$

It turns out that $[a|b|c|d] = 0$ just if the nim-sum $a \overset{*}{+} b \overset{*}{+} c \overset{*}{+} d = 0$, so Welter's Game with four coins can be played with a Nim-like strategy. To play with three coins, imagine a fourth coin on an extra square -1, and add one to each of the numbers labelling the squares while you calculate your move. E.g., {2,5,8} is like {0,3,6,9}, where the winning move would be to {0,3,5,6,}, so, in the three-coin position, move to {2,4,5}.

The mating method makes it easy to calculate the nim-value of a Welter position, but it's not so easy to find the good moves that make the nim-value zero. However, there's a remarkable connexion with *frieze patterns* [Conway and Coxeter 1973; WW, pp. 475–480], which work for nim-addition as well as for multiplication and ordinary addition, and which allow you (or your computer) both to calculate the value of the Welter function and to invert it.

Start with a row of zeros above the Welter position that you want to evaluate, and manufacture a frieze pattern (so called because, when it is extended to the right, it eventually repeats periodically) by completing diamonds

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array} \qquad \text{using the rule} \qquad a \overset{*}{+} d = (b \overset{*}{+} c) + 1,$$

so that $c = [a|d] \overset{*}{+} b$, where the sums are still nim-sums. Lo and behold (Figure 9) the value of the Welter function appears at the foot of the pattern, as follows from a formula on page 159 on ONAG.

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 2 & & 3 & & 5 & & 8 & & 13 & & 21 & \\
& & 2 & & 0 & & 5 & & 12 & & 4 & & 23 & & \\
& & & 3 & & 7 & & 13 & & 15 & & 31 & & & \\
& & & & 3 & & 12 & & 13 & & 11 & & & & \\
& & & & & 9 & & 13 & & 10 & & & & & \\
& & & & & & 15 & & 11 & & & & & & \\
& & & & & & & 14 & & & & & & &
\end{array}$$

Figure 9. Calculating the Welter function from a frieze pattern.

If you want to change the value $n = [a|b|c|\ldots]$ to some $n' \neq n$, then there are unique $a' \neq a$, $b' \neq b$, $c' \neq c$, … such that

$$[a'|b|c|\ \ldots] = n' = [a|b'|c|\ \ldots] = [a|b|c'|\ \ldots] = \ldots$$

and $[a|b|c|\ \ldots] = n$ remains true if we replace any *even* number of a, b, c, …, n by the corresponding primed letters. This *Even Alteration Theorem* [ONAG, pp. 160–162; WW, p. 477] may be written

$$\begin{bmatrix} a\,|\,b\,|\,c\,| \\ a'|b'|c'| \end{bmatrix}\!\!\cdots = \begin{matrix} n \\ n' \end{matrix}$$

To find $a', b', c', \ldots$ corresponding to a given n', continue the bottom row of the frieze pattern, n, n', n, n', n, $\ldots$ alternately, and then extend the pattern to the right, using the same diamond rule. You will discover that the defining row, a, b, c, $\ldots$ continues with the answers, a', b', c', $\ldots$!

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	5	8	13	21	15	0	37	35	10	11	19	
2	0	5	12	4	23	25	14	36	5	40	0	23		
3	7	13	15	31	24	25	41	5	15	45	29			
3	12	13	11	17	25	33	15	12	9	47				
9	13	10	6	31	46	4	7	11	8					
15	11	0	9	41	8	13	7	11						
14	0	14	0	14	0	14	0							

Figure 10. Inverting the Welter function using a frieze pattern.

In Figure 10 we find the good moves in the position {1,2,3,5,8,13,21} by choosing $n' = 0$ and extending the pattern of Figure 9. If you extend it even further to the right, you'll see why it's called a frieze pattern. If you believe the algorithm, and read the second row of Figure 10,

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 8 & 13 & 21 \\ 15 & 0 & 37 & 35 & 10 & 11 & 19 \end{bmatrix} = \begin{matrix} 14 \\ 0 \end{matrix}$$

Check that each move leads to a $\mathcal{P}$-position. Some of the suggested moves, e.g., 1 to 15, 3 to 37, are not legal, but, provided $n' < n$, you'll always find one that is legal, in fact there is always an odd number of legal good moves. Here there are three good moves: 2 to 0, 13 to 11, and 21 to 19.

We can even give you a strategy for the misère form (last player losing) of Welter's Game, if you're willing to learn about *Abacus Positions* [WW, pp. 478–481].

8. Coin-Turning Games

(See WW, pp. 429–456.) Several of the impartial games we've already mentioned, and a wide range of new games, can be realized by an idea of Hendrik Lenstra. The $\mathcal{P}$-positions in several of these turn out to correspond to the codewords in some well-known and some not-so-well-known error-correcting codes.

Turning Turtles was originally played with turtles, but it's less cruel to play it with a row of coins (Figure 11). A move is to turn a head to a tail, with the additional option of turning at most one other coin, to the left of it. This second

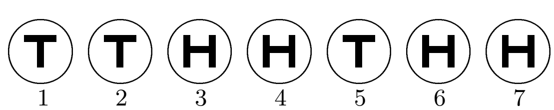

Figure 11. A Turning Turtles position, with coins 3, 4, 6, 7 showing heads.

coin may go from head to tail, or from tail to head. The game is over when all coins show tails, and the last player wins.

We leave you to verify that this is a disguise for Nim: if you number the coins 1, 2, 3, ... from the left, then the nim-value of coin n is $*n$ if it's a head, and 0 if it's a tail. The nim-value of a general position is the nim-sum of the nim-values, i.e., the nim-sum of the nim-values of the heads. For example, the good moves in Figure 11 are to turn coin 6 to a tail; or to turn 7 to a tail and 1 to a head; or to turn 4 to a tail and 2 to a head.

Mock Turtles is played in the same way, but a move may turn one, two or three coins, provided the rightmost turned goes from head to tail (this is to make the game satisfy the ending condition). We now number the coins from *zero* (the Mock Turtle) and find the nim-value (or Grundy function), $\mathcal{G}(n)$, of the n-th coin, when head up, to be:

$$\begin{array}{rl} n & = 0\ 1\ 2\ 3\ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ \ldots \\ \mathcal{G}(n) & = 1\ 2\ 4\ 7\ 8\ 11\ 13\ 14\ 16\ 19\ 21\ 22\ 25\ 26\ 28\ 31\ 32\ 35\ 37\ \ldots \end{array}$$

These are the *odious numbers* that we met as common values in Kayles.

$$\mathcal{G}(n) = 2n \text{ or } 2n+1.$$

To find which, write n in binary and append a check digit, 0 or 1, to make an *odd* number of digits 1.

Moebius, *Mogul* and *Moidores* are the corresponding games in which a move turns up to t coins, where $t = 5$, 7 and 9. We consider only odd values of t, because the *Mock Turtle Theorem* gives us the results for even values of t:

> Every nim-value for the $t = 2m+1$ game
> is an odious number.
> The corresponding value for the $t = 2m$ game
> is gotten by dropping the final binary digit.

The nim-values for coins 0 to 17 (when head-up) in Moebius are shown in Table 4. The structure of the $\mathcal{P}$-positions in 18-coin Moebius is revealed on replacing the coin numbers by the labels in the third row.

coin #	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
nim-value	1	2	4	8	16	31	32	64	103	128	171	213	256	301	342	439	475	494
label	∞	1	4	0	-4	-1	5	6	-8	2	-3	-5	8	3	-7	7	-6	-2

Table 4. Eighteen-coin Moebius gives the game its name.

Coins 0 to 5, with labels ∞, 0, ± 1, ± 4, clearly form a $\mathcal{P}$-position (whichever ones you turn over, I'll turn over the rest). Starting from this, or indeed from any $\mathcal{P}$-position, we can find others by operating on the labels with any *Möbius transformation* (modulo 17):

$$x \to \frac{ax+b}{cx+d} \quad \text{with} \quad ad - bc = 1.$$

There are	1 +	102 +	153 +	153 +	102 +	1	$\mathcal{P}$-positions
with respectively	0	6	8	10	12	18	heads.

If we drop the Mock Turtle (at ∞) we have the $t = 4$ game on 17 coins. The $\mathcal{P}$-positions in these two games correspond to the codewords in the [18,9,6] extended quadratic residue code and the [17,9,5] quadratic residue code.

Similarly if we play 24-coin Mogul ($t = 7$, turn up to 7 coins) we find

	1 +	759 +	2576 +	759 +	1	$\mathcal{P}$-positions
with	0	8	12	16	24	heads

coinciding with the 2^{12} codewords of the extended [24,12,8] Golay code. With $t = 6$ and 23 coins the $\mathcal{P}$-positions correspond to the codewords in the perfect [23,12,7] Golay code. Robert Curtis [1976; 1977] has given a pictorial representation of this, the Miracle Octad Generator or "MOG", which also shows the connexion with the Steiner system $S(5, 8, 24)$.

In the *Ruler Game* any number of *contiguous* coins may be turned (with the rightmost always going from head to tail). If the coins are numbered from 1, then the nim-value, $\mathcal{G}(n)$, is the highest power of 2 that divides n.

In *Turnips* (or *Ternups*) a move turns three equally spaced coins. Number the coins from 0 and write n in *ternary* (base 3). Then $\mathcal{G}(n)$ is the k-th odious number if the last digit 2 in the ternary expansion is in the k-th place from the right, or $\mathcal{G}(n) = 0$ if there is no digit 2 in the ternary expansion of n.

There is a plethora of such coin-turning games. They can also be played on a two-dimensional array of coins. For example, we can play the Cartesian product, $A \times B$, of two one-dimensional games A and B, in which a move is to turn all coins with coordinates (a_i, b_j), where $\{a_i\}$ and $\{b_j\}$ are sets of coins constituting legal moves in games A and B respectively. To satisfy the ending condition, the "most northeasterly" coin turned must go from head to tail (Figure 12).

Figure 12. A typical move in Moebius $\times$ Turnips.

The nim-value of a (head-up) coin in such a game is given by the *Tartan Theorem*:

The nim-values for the game $A \times B$ are the nim-products of those for A and B: $\mathcal{G}_{A\times B}(a,b) = \mathcal{G}_A(a) \overset{*}{\times} \mathcal{G}_B(b)$

where $\mathcal{G}_A(a)$ is the nim-value of coin number a in game A, etc., and $\overset{*}{\times}$ denotes *nim-multiplication*. Nim-multiplication [ONAG, pp. 52–53; Lenstra 1977] may be defined from the field laws (e.g., associativity and distributivity over nim-addition), together with the rule

$$n \overset{*}{\times} N = n \times N \quad \text{for} \quad n < N$$
$$N \overset{*}{\times} N = 3N/2$$

where N is any *Fermat power* of 2 $(2, 4, 16, \ldots, 2^{2^h}, \ldots)$. For example, $2 \overset{*}{\times} 2 = 3$, because 2 is a Fermat power, while $2 \overset{*}{\times} 3 = 2 \overset{*}{\times} (2 \overset{*}{+} 1) = 3 \overset{*}{+} 2 = 1$. A more complicated example is

$$\begin{aligned} 13 \overset{*}{\times} 7 &= (4 \overset{*}{\times} 3 \overset{*}{+} 1) \overset{*}{\times} 7 = (4 \overset{*}{\times} (2 \overset{*}{+} 1)) \overset{*}{\times} (4 \overset{*}{+} 2 \overset{*}{+} 1) \overset{*}{+} 7 \\ &= 4 \overset{*}{\times} 4 \overset{*}{\times} (2 \overset{*}{+} 1) \overset{*}{+} 4 \overset{*}{\times} 2 \overset{*}{\times} (2 \overset{*}{+} 1) \overset{*}{+} 4 \overset{*}{\times} (2 \overset{*}{+} 1) \overset{*}{+} 7 \\ &= 6 \overset{*}{\times} (2 \overset{*}{+} 1) \overset{*}{+} 4 \overset{*}{\times} (3 \overset{*}{+} 2) \overset{*}{+} 4 \overset{*}{\times} 3 \overset{*}{+} 7 \\ &= (4 \overset{*}{+} 2) \overset{*}{\times} (2 \overset{*}{+} 1) \overset{*}{+} 4 \overset{*}{\times} 1 \overset{*}{+} 12 \overset{*}{+} 7 \\ &= 4 \overset{*}{\times} 2 \overset{*}{+} 2 \overset{*}{\times} 2 \overset{*}{+} 4 \overset{*}{+} 2 \overset{*}{+} 4 \overset{*}{+} 11 \\ &= 8 \overset{*}{+} 3 \overset{*}{+} 9 \\ &= 2. \end{aligned}$$

The assiduous reader will verify that the nim-cube-roots of 1 are 1, 2 and 3, and the nim-fifth-roots are 1, 8, 10, 13, 14.

9. Misère Nim and an Awful Warning

The power of the Sprague-Grundy theory derives from its reduction of all impartial games to the game of Nim. But particular games may not break up naturally into disjunctive sums, so that much of the force is lost. There are other reasons why it may be difficult or tedious or *hard* in the technical sense, to calculate the nim-value of a position. Also, the theory applies only to normal play.

When Bouton gave his analysis of Nim, he also noted that only a small modification is needed to cover the *misère* version, in which the last player *loses*. To win misère nim, play just as in ordinary Nim until all the heaps, with just one exception, contain a single bean. Then take all the beans from the exceptional heap, or all the beans but one, so as to leave an odd number of heaps of size one.

It's tempting to think (and several people have been tempted to write) that you can play misère impartial games just like normal impartial games until very near the end, when you ...

BUT THAT'S NOT TRUE!

The situation is very complicated. The little that is known in general is given in WW, Chapter 13. However, an intriguing breakthrough was made recently by William Sibert and John Conway [1992], who have given an analysis of *Misère Kayles* and Thane Plambeck [1992] has used their method to analyze a small number of other games.

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